The de Rham Homomorphism

Andrew Tindall
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1 Introduction

Our aim is to describe a homomorphism from one type of cohomology to another: That is, given a C^∞ manifold M, and some integer p, to construct a homomorphism ℓ from the pth de Rham cohomology group $H^p_{dR}(M)$ to the pth singular cohomology group with coefficients in $\mathbb{R}$: $H^p(M;\mathbb{R})$. Having done so, we will then show that this homomorphism is natural: that, given a smooth map $F : M \to N$ of manifolds, the following diagram commutes:

$$\begin{tikzcd}
H^p_{dR}(N) \arrow[r, "F^*"] \arrow[d, "\ell"] & H^p_{dR}(M) \arrow[d, "\ell"] \\
H^p(N;\mathbb{R}) \arrow[r, "F^*"] & H^p(M;\mathbb{R})
\end{tikzcd}$$

where F^* is the map induced by F on either of the respective cohomologies.

2 Definition of the de Rham cohomology of a manifold

Let M be a C^∞ manifold. We define the *covariant k-tensor bundle* $T^kT^*(M)$ as follows:

$$T^kT^*(M) = \coprod_{p\in M} T^k(T^*_pM),$$

Where $T^*_p(M)$ is the cotangent space to M at a point p, and $T^kT^*(V)$ is the vector field of covariant k-tensors - that is, k-multilinear maps $M \to \mathbb{R}$. We then define the smooth subbundle of alternating k-forms, $\Lambda^kT^*(M)$, as follows:

$$\Lambda^kT^*(M) = \coprod_{p\in M} \Lambda^k(T^*_pM),$$

where the space $\Lambda^kT^*(V)$ of alternating k-forms on a vector space V is the space of k-multilinear functions $V \to \mathbb{R}$ which are *alternating* in their arguments - i.e. for any function $\phi \in \Lambda^kT^*(V)$, and arguments $v_1, \cdots, v_k$, we have

$$\phi(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -\phi(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

Thus, given a basis $x_1, \cdots, x_n$ of V, with $dx_1, \cdots, dx_n$ the corresponding dual basis, any ϕ in $\Lambda^k T^*(V)$ can be uniquely written as

$$\phi = \sum_{i_1 < i_2 < \cdots < i_k} \phi_{i_1, i_2, \cdots, i_k} \cdot dx_{i_1} \wedge dx_{i_2} \wedge \cdots, \wedge dx_{i_k};$$

where each $\phi_{i_1, \cdots, i_k}$ is a C^∞ function $V \to \mathbb{R}^n$, and $\wedge$ is the normal alternating wedge product ([1], p. 15, and [3], pp. 354 - 5). From now on we will shorten an arbitrary k-tuple of indices $i_1, \cdots, i_k$ as I, and write $dx_{i_1} \wedge dx_{i_k}$ as dx_I. On $\mathbb{R}^n$, the standard dual basis for $T^*(\mathbb{R}^n)$ is the set of coordinate projection dx_i; on our arbitrary smooth manifold M, after passing to a local coordinate system, we can use the same dual basis dx_i of the cotangent space $T_p^*(M)$.

Having defined the bundle of alternating k-tensors $\Lambda^k T^*(M)$ pointwise, a section of this smooth vector bundle is a function $f : M \to \Lambda^k T^*(M)$ with the following properties: $f(p) \in \Lambda^k(T_p^*(M))$ for all $p \in M$, and, for every coordinate system (ϕ, U), the function $\overline{f} : \phi^{-1}(U) \to \Lambda^k T^*(\mathbb{R}^n)$ is represented by $\binom{n}{k}$ smooth functions; i.e. locally,

$$\overline{f} = \sum_I f_I dx_I,$$

where each f_I is a smoothly varying function from $\phi^{-1}(U)$ to the space of C^∞ functions $\mathbb{R}^n \to \mathbb{R}$. We call such a section a *differential k-form*, and denote the space of such k-forms by

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M).$$

We equip this space with the *exterior derivative* operator d, which takes k-forms to $k+1$-forms by the following formula (in local coordinates):

$$\begin{aligned} d\left(\sum_I f_I dx_I\right) &= \sum_I df_I dx_I \\ &= \sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I. \end{aligned}$$

On $\mathbb{R}^n$, this is indeed a function $\Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$; to extend this definition to an arbitrary manifold M, we first need to see that forms, as sections of covariant bundles, can be pulled back along arbitrary smooth maps: i.e., for a form $f \in \Omega^k(N)$, and a C^∞ map of manifolds $\alpha : M \to N$, we have a k-form $\alpha^*(f) \in \Omega^k(M)$, defined pointwise on M by, for $p \in M$, by

$$\alpha^*(f)(p) = \sum_I (f_I \circ \alpha) d\alpha_I,$$

where

$$f_{\alpha(p)} = \sum_I f_I dx_I,$$

and

$$d\alpha_I = (dx_{i_1} \circ d\alpha_p) \wedge \cdots \wedge (dx_{i_k} \circ d\alpha_p),$$

where $d\alpha_p : T_pM \to T_{\alpha(p)}N$ is the differential of the map α. This makes each $dx_i \circ d\alpha_p$ a linear function $T_pM \to \mathbb{R}$, and so $\langle dx_i \circ d\alpha_p \rangle_{1 \leq i \leq n}$ is a basis of (a subspace of) T_p^*M. So, indeed, $\alpha^*(f)(p)$ is an element of $\Lambda^k T_p^* M$, and the whole map $\alpha^*(f)$ takes local sections of $\Lambda^k T^* N$ to local sections of $\Lambda^k T^* M$. Now, we can use this to define the differential operator on an arbitrary smooth manifold. First, restrict to some open set U with a local chart (ϕ, U), and define the differential of a k-form f on U by

$$df = \phi^*(d(\phi^{-1*}(f))).$$

We need to see that this map is in fact indpendent of coordinates; this follows from the fact that the pullback on forms is contravariant functorial (shown in [3], p. 365), and that the exterior derivative on $\mathbb{R}^n$ and $\mathbb{R}^m$ commutes with pullbacks $\mathbb{R}^n \to \mathbb{R}^m$ (shown in [3], p. 364): Let (U, ϕ) and (V, ψ) be two charts, so that $\phi \circ \phi^{-1}$ is a diffeomorphism of subsets of $\mathbb{R}^n$; and let f be a k-form on $U \cap V$:

$$\begin{aligned}
(\phi \circ \psi^{-1})^*(d(\phi^{-1*}(f))) &= d((\phi \circ \psi^{-1})^*(\phi^{-1*}(f))) \\
&= d(\psi^{-1*}\phi^*\phi^{-1*}(f)) \\
&= d(\phi^{-1*}(f)).
\end{aligned}$$

Since $\phi^{-1*}(f)$ is an element of $\Omega^k(\mathbb{R}^n)$, we can use the coordinatewise definition of the exterior derivative; taking the exterior derivative and pulling back along ϕ, we are left with an element of $\Omega^{k+1}(U)$. Since the exterior derivative respects coordinate transformations, it is a well-defined map of global sections of $\Lambda^k T^*(M)$ to sections of $\Lambda^{k+1}T^*(M)$. In short, we have a map $\alpha^* : \Omega^k(M) \to \Omega^{k+1}(M)$.

We thus have a sequence of vector spaces:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1} \xrightarrow{d} \cdots$$

In fact, this is a chain complex: the composition $d \circ d$ is equal to zero - this boils down, in local coordinates, to the fact that mixed partial derivatives are equal, while the wedge product is anticommutative ([1], p. 15). So in fact this is a cochain complex of $\mathbb{R}$-vector spaces, and we can take its cohomology: the kth cohomology group of this complex, $H^k_{dR}(M)$, is the quotient $\ker(d|_{\Omega^k})/\mathrm{im}(d|_{\Omega^{k-1}})$. We refer to $\ker(d|_{\Omega^k})$ as *closed* k-forms and $\mathrm{im}(d|_{\Omega^{k-1}})$ as *exact* k-forms. Thus, the k-th cohomology group (really a vector space over $\mathbb{R}$) is the quotient of closed k-forms by exact k-forms; This is the *de Rham cohomology* of the manifold M.

3 Singular Cohomology

We will now construct another, apparently different cohomology theory on a manifold M: the singular cohomology with coefficients in $\mathbb{R}$. The basic objects of singular homology are the standard simplices Δ_p:

$$\Delta_p = \left\{ \langle x_0, \cdots, x_n \rangle \in \mathbb{R}^{n+1} \mid 0 \leq x_i \leq 1, \sum x_i = 1 \right\}.$$

A singular p-simplex in M is a continuous map $\Delta_p \to M$, and a singular p-chain is a finite formal linear combination of p-simplices. Denote by $C_p(M)$ the free abelian group of p-chains generated by all the singular p-simplices in M.

For any simplex Δ_p, with $p > 0$, the ith face is an inclusion $F_{i,p} : \Delta_{p-1} \to \Delta_p$, which takes the $p-1$ simplex to the i-th face of Δ_p (the set of vectors $\langle x_0, \cdots, x_n \rangle$ in Δ_p whose x_i component is 0). For any $p > 0$, the *boundary map* $\partial : C_p(M) \to C_{p-1}(M)$ is defined on each singular $p+1$-simplex by

$$\partial(\sigma) = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}.$$

It can be quickly seen that $\partial \circ \partial = 0$; essentially because any simplex of codimension 2 in Δ_p is counted twice, with opposite signs, when taking the boundary of the bounday of Δ_p. ([2], p. 105) Extending this definition of ∂ to the whole free abelian group C_p, we have a chain complex

$$\cdots \xrightarrow{\partial} C_p \xrightarrow{\partial} C_{p-1} \xrightarrow{\partial} \cdots \qquad C_1 \xrightarrow{\partial} C_0 \ ;$$

The homology of this complex is the singular cohomology of M with coefficients in $\mathbb{R}$. In fact, by the universal coeffcient theorem, we need only have taken the singular homology over $\mathbb{Z}$ ([2], p. 198). However, we are here interested in the cohomology: let $\mathrm{Hom}(\cdot, \mathbb{R})$ be the contravariant Hom-functor: for each C_p we define the $\mathbb{R}$-vector space

$$C^p = \mathrm{Hom}(C_p, \mathbb{R}),$$

and by functoriality, we have the dual map induced by ∂:

$$d : \mathrm{Hom}(C_p, \mathbb{R}) \to \mathrm{Hom}(C_{p+1}, \mathbb{R}).$$

These assemble into a cochain complex

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^p \xrightarrow{d} C^{p+1} \xrightarrow{d} \cdots .$$

The cohomology of this complex is the *singular cohomology of M with coefficients in M*: the pth singular cohomology group is defined as

$$H^p(M; \mathbb{R}) = \ker(d|_{C^p}) / \mathrm{im}(d|_{C^{p-1}}).$$

It at first appears that this definition must be insufficient to describe the smooth structure of the manifold M, since the singular maps involved are merely continuous. There is in fact a definition of a smooth singular homology; it is identical, except that each simplex $\sigma : \Delta_p \to M$ is required to be smooth (in the sense that it has a smooth extension near each point - see [3], p. 473). We denote by C_p^∞ the *smooth chain group*; we then construct a chain complex with the same methods. However, the resulting homology is in fact isomorphic to the singular homology ([3], p. 474). The upshot is that we can pick a smooth representative c for any homology class $[c]$ of a singular chain in M, and that we need only define elements in $H^p(M; \mathbb{R})$ on smooth chains, since any class will always have such a representative.

4 The de Rham homomorphism

The de Rham theorem is a powerful statement linking the de Rham cohomology of a manifold and its singular cohomology with coefficients in $\mathbb{R}$. In fact, it is really a step towards the de Rham Theorem, which states that the homomorphism constructed here is an isomorphism. However, (in my opinion) it is surprising enough that the two cohomologies are related at all: they are constructed in two almost-opposite directions; $H_{dR}(M)$ from the starting point of smooth tensor fields and differential forms, and $H(M;\mathbb{R})$ from the point of simplices and maps which are only continuous, not necessarily smooth.

The de Rham homomorphism will be a function ℓ from the k-th cohomology group of one complex to the k-th cohomology group of the other; this map will turn out to be not only a homomorphism, but natural, in that it commutes with pullbacks. Before constructing this map, we need the following machinery:

Let M be a smooth manifold, f a closed p-form on M, and σ a smooth p-simplex in M (a map $\Delta_p \to M$ which can be extended to a smooth map in a neighborhood of Δ_p). Define the integral of f over σ to be

$$\int_\sigma f = \int_{\Delta_p} \sigma^* f.$$

This is a sensible definition because the pullback $\sigma^* f$ is a p-form on Δ_p, which is a smooth p-submanifold of $\mathbb{R}^p$ which inherits orientation from $\mathbb{R}^p$ ([3], p. 481). Extend this to a smooth p-chain $c = \sum_{i=1}^k c_i \sigma_i$ by

$$\int_c f = \sum_{i=1}^{k} \int_{\sigma_i} f.$$

Now, for a smooth simplex σ, our results on p-forms and the differential operator, combined with Stokes' theorem on manifolds ([3], p. 411) gives the following result:

$$\begin{aligned} \int_\sigma df &= \int_{\Delta_p} \sigma^*(df) \\ &= \int_{\Delta_p} d(\sigma^* f) \\ &= \int_{\partial\Delta_p} \sigma^* f. \end{aligned}$$

The boundary $\partial\Delta_p$ is the sum of faces $F_{i,p}$, with orientation either preserved or reversed, depending on whether i is even ([3], p. 481). The result is that

$$\int_\sigma df = \sum_{i=0}^{p} (-1)^i \int_{\sigma \circ F_{i,p}} f.$$

Extending this linearly to any p-chain c, we have *Stokes' Theorem for Chains*:

$$\int_{\partial c} f = \int_c df.$$

Now, we can construct the map $\ell : H^p_{dR}(M) \to H^p(M;\mathbb{R})$. First, take some cohomology class $[f]$ of closed k-forms on M, represented by the closed form f. We want to construct an element of $H^p(M;\mathbb{R})$; these are represented by maps $C_p(M;\mathbb{R}) \to \mathbb{R}$, so we first construct such a map. Let $\sigma : \Delta_p \to M$ be a smooth p-simplex on M (as mentioned at the end of the last section, for the purposes of constructing an element of $H^p(M;\mathbb{R})$ we may assume that any singular simplex is smooth.) Define $\ell(f)(\sigma)$ by

$$\ell(f)(\sigma) = \int_\sigma f;$$

extending $\mathbb{R}$-linearly, we have a map $C_p(M;\mathbb{R}) \to \mathbb{R}$. This is a well-defined element of $H^p(M;\mathbb{R})$: let c, c' be chains in the same homology class; then $c - c' = \partial b$ for some $p+1$-chain b, and we see that $\ell(f)$ is zero on $c - c'$:

$$\begin{aligned}
\ell(f)(c-c') &= \ell(f)(\partial b) \\
&= \int_{\partial b} f \\
&= \int_b df \\
&= \int_b 0 \\
&= 0,
\end{aligned}$$

where $df = 0$ because $f \in \ker(d)$.

Further, this is a well-defined map on cohomology classes of closed k-forms: if f and g represent the same cohomology class, then $f - g = dh$ for some $p-1$ form h, and for any smooth simplex σ we have

$$\begin{aligned}
\ell(f-g)(\sigma) &= \ell(dh)(\sigma) \\
&= \int_\sigma dh \\
&= \int_{\partial\sigma} h \\
&= \int_0 h \\
&= 0
\end{aligned}$$

where $\partial\sigma = 0$ because σ is a chain, and thus has 0 boundary. We have therefore constructed our desired homomorphism:

$$\ell : H^p_{dR}(M) \to H^p(M;\mathbb{R}).$$

This homomorphism respects most all of the deeper structural properties of both cohomology theories. Firstly, given a smooth map of manifolds $F : M \to N$, we have two collections of induced pullback maps, which by a grand abuse of notation are all denoted F^*, from one cohomology group on N to the corresponding group on M. We see that ℓ commutes with these pullbacks; i.e. that the following diagram commutes:

$$\begin{array}{ccc} H^p_{dR}(N) & \xrightarrow{F^*} & H^p_{dR}(M) \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle \ell} \\ H^p(N;\mathbb{R}) & \xrightarrow{F^*} & H^p(M;\mathbb{R}) \end{array}$$

To see this, we first see that, for a smooth p-simplex σ on M and a smooth p-form f on N, we have

$$\begin{aligned} \int_\sigma F^* f &= \int_{\Delta_p} \sigma^* F^* f \\ &= \int_{\Delta_p} (F \circ \sigma)^* f \\ &= \int_{F\circ\sigma} f. \end{aligned}$$

Further, the class of the map $F \circ \sigma$ in $H_p(N;\mathbb{R})$ is exactly the pushforward $F_*(\sigma)$ of the class $[\sigma]$ in $H_p(M;\mathbb{R})$. And, by the definition of the pullback F^* on singular cohomology, for any $g \in H^p(N;\mathbb{R})$, we have

$$F^*(g)(\sigma) = g(F_*(\sigma)).$$

Putting all these together, we have

$$\begin{aligned} \ell(F^*(f))(\sigma) &= \int_\sigma F^* f \\ &= \int_{F\circ\sigma} f \\ &= \ell(f)(F \circ \sigma) \\ &= \ell(f)(F_*(\sigma)) \\ &= F^*(\ell)(\sigma) \end{aligned}$$

Which was to be shown. So indeed ℓ commutes with pullbacks.

5 the de Rham Theorem

That ℓ commutes with pullbacks is only one half of the naturality argument essential to proving the de Rham theorem; to fully prove it as in [3], we would first need to construct the Meyer-Vietoris sequence for both cohomologies, and then show that ℓ also commutes with the two respective connecting morphisms δ and ∂^*. We would then reduce to the case of a finite cover of M, and by induction, assume that we have covered it with two sets U and V on which ℓ is an isomorphism. Then ℓ induces a natural map between the two Meyer Vietoris sequences, which is an isomorphism on the direct sum terms and intersection terms; therefore, by the five lemma, it is an isomorphism on each cohomology group of M.

References

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