Stone Spaces & Profinite Groups

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1 Profinite Spaces, Topologically

The study of profinite spaces, also called *Stone Spaces*, arises naturally from certain fields of algebra: many topological groups are profinite, and *profinite groups* are worth studying in their own right. However, we can first discuss the topological definition and properties of these spaces. We begin by defining two terms:

Definition 1.1. A topological space is X *quasi-compact* if, for any open cover $X = \bigcup_{i \in I} U_i$ of X where I may be infinite, there is some finite subcover $X = \bigcup_{i=1}^{n} U_i$.

Note that this is precisely the definition of a *compact space* given in many topology textbooks, including, for example, [1]. However, in other fields, such as algebraic geometry, the definition of a *compact* topological space is one which is both quasi-compact and Hausdorff. This distinction, and the prevalence of quasi-compactness in sources on Stone spaces, is why we have a new term for a property which would, in other contexts, be called by another name.

Now, another definition is needed before we define profiniteness:

Definition 1.2. A topological space X is *totally disconnected*[1] whenever the only connected nonempty subsets are the singletons.

Here are some examples to help illustrate this property:

- Every discrete space is totally disconnected, since any subset with more than two elements x and y will necessarily contain the non-trivial clopen sets $\{x\}$ and $\{y\}$,
- The Cantor set is totally disconnected. Recall that the Cantor set is the subset of $[0, 1] \subset \mathbb{R}$ formed by removing first the middle-third $(1/3, 2/3)$ of the interval, then the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals, and continuing as such, leaving only those points which do not lie in any of the removed intervals. Equivalently, it is the set of all rational numbers between 0 and 1 which may be represented in ternary without using the digit 1.
- The rational numbers $\mathbb{Q}$ are totally disconnected.
- For any prime number p, the p-adic integers $\mathbb{Z}_p$, with the p-adic norm topology are totally disconnected.

[1] In french, somewhat vexingly, such a space is *totalement discontinu*.

A totally disconnected space, in a sense, "looks discrete." However, it does not necessarily have the discrete topology, as the latter three examples above show.

Now, we can define a certain class of topological spaces:

Definition 1.3. A *profinite topological space* is a space X such that

- X is Hausdorff,
- X is quasi-compact, and
- X is totally disconnected.

We note that any finite discrete space is profinite, while any infinite discrete space is not, as it fails the quasicompactness hypothesis. The rational numbers are not profinite, as they also fail the quasi-compactness hypothesis. However, the Cantor space is profintie: it inherits the Hausdorff property from $\mathbb{R}$, and it is the intersection of the closed sets $[0,1], ([0,1/3] \cup [2/3,1]), \ldots,$ and is therefore a closed and bounded subset of $\mathbb{R}$, making it quasi-compact (recall that this means it is "compact" in the familiar sense of the word). Finally, the space $\mathbb{Z}_p$ is, as it will turn out, profinite.

Thus far, the necessary definitions have been relegated to the realm of point-set topology. However, there is another equivalent definition of a profinite space, which is perhaps less intuitive, but which is invaluable in the study of profinite *groups*. In order to state this second definition, we now take a brief detour into the world of category theory.

2 Profinite Spaces, Categorically

The machinery of inverse limits we are about to define will allow us to state that profinite spaces are, in a sense, "built from" finite discrete spaces.

Recall that a category is a collection of objects and composable morphisms between those objects. The categories at hand are concrete, so we may think of an object as a set, perhaps with some structure, and a morphism as a set map which preserves whatever extra structure our category requires. In particular, we will work in the category **Top** of topological spaces and continuous maps, and later the category **TGrp** of topological groups and continuous group homomorphisms.

Definition 2.1. A *filtered diagram* in a category C is a collection of objects $\{A_i\}_{i\in I}$ (indexed over any set I) which satisfy the following "filtered" conditions:

(a) For any pair A_i and A_j of objects, there is some third object A_k, with two morphisms $f_{ki} : A_k \to A_i$ and $f_{kj} : A_k \to A_j$.

(b) For any pair A_i and A_j of objects, and any pair of arrows $f_1 : A_i \to A_j$ and $f_2 : A_i \to A_j$, there is a third object A_k, with a morphism $g : A_k \to A_i$ such that $f_1 \circ g = f_2 \circ g$.

The first requirement can be profitably thought of as any pair of objects having at least one "common lower bound," and the second one as stating that any pair of arrows is "co-eventually equal." But without worrying so much about those details, it will be useful to have the following simple picture of a cofiltered diagram:

$$\cdots \longrightarrow A_i \longrightarrow A_j \longrightarrow A_k \longrightarrow A_l.$$

Now, given such a diagram, the limit of the diagram is an object satisfying a certain universal property.

Definition 2.2. Let $\{A_i\}_{i \in I}$ be a diagram as above (the cofiltered property will not be a necessary hypothesis for the definition of a limit). Then an object A is the *limit* of this diagram if, for each A_i in the diagram, we have a unique map $\varphi_i : A \to A_i$, such that the φ_i are compatible with the diagram maps; that is, for any $f_i : A_i \to A_j$, we have $\varphi_j = f_i \circ \varphi_i$, and A is universal with respect to this property: if we have any other object B, with maps ψ_i that are likewise compatible with the maps of the diagram, then we have a unique map $g : B \to A$ such that, for all i, $\psi_i = \varphi_i \circ g$.

Again, the particulars of this definition are not worth keeping in mind, so much as the following diagrams: first, here is an illustration of the maps from A to the various objects A_i of the diagram. The important thing is that all possible paths from A to an object of the diagram should be equal, so that this diagram is *commutative*.

$$\begin{tikzcd}
 & & A \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_i \arrow[r] & A_j \arrow[r] & A_k \arrow[r] & A_l
\end{tikzcd}$$

If this has sunk in, consider another object B with the same properties as A. Then we have a very similar diagram, with B lying over $\{A_i\}_{i \in I}$ instead:

$$\begin{tikzcd}
 & & B \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_i \arrow[r] & A_j \arrow[r] & A_k \arrow[r] & A_l
\end{tikzcd}$$

Then the *universal property of the limit* states that there must be a *unique* map from B to A, so that we can first go to A from B, then follow the arrows to the objects.

$$\begin{tikzcd}
 & & B \arrow[ddll] \arrow[ddl] \arrow[d] \arrow[dd, bend right] \arrow[ddr] \arrow[ddrr] & & \\
 & & A \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_i \arrow[r] & A_j \arrow[r] & A_k \arrow[r] & A_l
\end{tikzcd}$$

In essence, we can clear up the above diagram: if we start at B, we can always pass through A before going to any other object:

$$\begin{tikzcd}
 & & B \arrow[d] & & \\
 & & A \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\cdots \arrow[r] & A_i \arrow[r] & A_j \arrow[r] & A_k \arrow[r] & A_l
\end{tikzcd}$$

This last diagram is a good image to have for a limit in any category.

In fact, there is another, more concrete way to characterize the limit. If we let $\prod_{i \in I} A_i$ be the product of all the objects in our diagram, that is, the set of all "lists," "tuples," or "sequences" of elements $(a_i)_{i \in I}$ (although all these words might imply that I be countable, which it need not be), then the limit of the diagram is a certain subset of $\prod_{i \in I} A_i$.

Lemma 2.3. The limit of the diagram $\{A_i\}_{i\in I}$ is isomorphic to the subset of the product of all objects in the diagram, consisting of all those elements of the product $(a_i)_{i\in I}$, where $(a_i)_{i\in I}$ is compatible with each of the maps f_i in the diagram: that is, for each two objects A_i and A_j and any map $f : A_i \to A_j$, we have $f(a_i) = a_j$.

Note that the product is the *categorical* product, so that, in the case of the category **Top**, it is also imbued with the product topology.

Intuitively, any element in A must map "down to" each of the objects in A_i, giving us a list of elements $(a_i)_{i\in I}$; the fact that all the maps in question commute tells us that $f_i(a_i) = a_j$ for any f_i; and because A is totally determined by how it maps into the diagram, this is enough information to tell us all of the elements of A.

Now that we know a little bit about cofiltered diagrams and the categorical limit, we can state another definition of the word "profinite:"

Definition 2.4. A *profinite space* is the limit of a cofiltered diagram which consists entirely of finite sets, with the discrete topology, and continuous maps between those finite sets.

At first glance, the two definitions we have for the word *profinite* are not apparently related, but in fact they are:

Theorem 2.5. The two definitions **??** and **??** of a profinite space are equivalent. That is, a topological space is quasi-compact, Hausdorff, and totally disconnected if and only if it is the limit of a cofiltered diagram of finite sets.

The proof of Theorem **??** is somewhat canonical; we will follow the steps from the Stacks Project at [**?**, Lemma 08ZY].

First, we show that Definition **??** implies Definition **??**. Let $\{A_i\}_{i\in I}$ be some cofiltered diagram of finite, discrete topological spaces, and let A be its limit, which by Lemma **??** is a subspace of the product $\prod_{i\in I} A_i$. We wish to show that A is a profinite topological space.

We first show that it is Hausdorff. If a and b are distinct points in A, then they must be distinguishable somehow: they map to different elements a_i and b_i in some object A_i of our diagram. Then A_i is a finite discrete space, so $\{a_i\}$ and $\{b_i\}$ are both open sets, and a is contained in the open set $\varphi_i^{-1}(\{a_i\})$, and b contained in the disjoint open set $\varphi^{-1}(\{b_i\})$. So A is Hausdorff.

Now, we show that A is totally disconnected. Again, let a and b be distinct elements of A, mapping to some a_i and b_i in A_i. It is a fact that, if two elements a_i and b_i are in different connected components in A_i, then any two elements of the product $\prod_{i\in I} A_i$ having a_i and b_i, respectively, as their ith components, must also be in different connected components in $\prod_{i\in I} A_i$. So, any two elements a and b are in different connected components of A, meaning that the only connected components of A are the singleton sets, and it is totally disconnected.

Finally, we note that A is quasi-compact. In fact, any limit of compact spaces is quasi-compact: the product of any collection of quasi-compact spaces is quasi-compact, and the subset of elements $(a_i)_{i\in I}$ compatible with the maps of the given diagram is a closed subset: if A_i and A_j are two objects, and $f : A_i \to A_j$ is a morphism between them, then let $\Gamma_f \subset A_i \times A_j$ be the graph of f: $\Gamma_f = \{(a_i, a_j) \mid f_i(a_i) = a_j\}$. Then Γ_f is closed, since it is the graph of a continuous function, and A is the intersection of all the following closed subsets of $\prod_{i\in I} A_i$, for every f in the diagram:

$$\Gamma_f \times \prod_{k\neq i,j} A_k \subset \prod_{i\in I} A_i.$$

The upshot is that A is a closed subset of a quasi-compact space, and therefore quasi-compact itself. So, we have finished showing that A is profinite.

Now, we show that definition **??** implies definition **??**. This proof is a little subtler. Let A be a quasi-compact, Hausdorff, totally disconnected space, and let $\mathcal{I}$ be the set of decompositions of A into finite, disjoint unions: $A = \coprod_{i \in I} U_i$, with U_i a nonempty open set. That is: every element in $\mathcal{I}$ corresponds to a particular way of decomposing A as a finite number of nonempty, disjoint open sets.

Then, if $I \in \mathcal{I}$ is the decomposition $A = \coprod_{i \in I} U_i$, let $g : A \to I$, where the finite set I has the discrete topology, be the map sending any element of U_i to i. So, all the elements of the first set U_k are sent to $k \in I$, the elements in U_j to j, and so on. This map g is *continuous*: the inverse image of any open singleton $\{k\}$ of I is U_k, which is open by definition.

We can now define a cofiltered structure on $\mathcal{I}$. First, a definition:

Definition 2.6. We say that a decomposition $A = \coprod_{j \in J} U_j$ *refines to* a decomposition $A = \coprod_{i \in I} U_i$ whenever every open set U_i in the second cover is a subset of some U_j in the first cover.

Then, for any two such coverings corresponding to the finite sets J and I, we have a map $I \to J$, defined by sending $i \in I$ to $j \in J$ whenever $U_i \subset U_j$. Since I and J have the discrete topology, this is automatically a continuous map of finite spaces. And, if we have any two open covers corresponding to I and J, then there is an open cover corresponding to some set $K \subset I \times J$, defined by taking only the nonempty sets in the following decomposition:

$$A = \coprod_{(i,j) \in I \times J} U_i \cap U_j.$$

This cover is a refinement of the covers corresponding to both I and J, so we have a map $K \to I$ and $K \to J$. Therefore, we can assemble all of the finite sets corresponding to decompositions in $\mathcal{I}$ into a cofiltered diagram of finite sets with the discrete topology:

$$\cdots \longrightarrow L \longrightarrow K \longrightarrow J \longrightarrow I \longrightarrow *,$$

where this final element is the decomposition $A = \coprod_{i=*} A$.

We have a cofiltered diagram of finite sets, so we may take its limit: let $A' = \prod_{I \in \mathcal{I}} I$ be the limit of the above diagram. We have a map $A \to A'$, sending any point x to i in I if it is contained in the open set U_i in the decomposition $A = \prod_{i \in I} U_i$.

In fact, this map is a homeomorphism. It is continuous, because it is a product of maps into discrete spaces, which are necessarily continous. And it is surjective: any element of A' is defined by a collection of clopen subsets U_i of A, with $U_i \subset U_j$ whenever we have a map $U_i \to U_j$; then any finite collection of these subsets has some common set U_k contained in each of them (by the cofiltered property), and therefore any finite collection of these subsets is nonempty. By [**?**, Lemma 005D], this means that the intersection of the entire collection of these subsets is nonempty, containing some element x which maps to each $i \in I$ corresponding to this element of A'.

We also see that the is injective, because A is totally disconnected, and $\{x\}$ is the intersection of all clopen subsets of A containing x ([**?**, Lemma 08ZN]), meaning that there is only one x which may map to any given element of A.

So, we have shown that the two definition of profinite are, in fact, equivalent.

3 Profinite Groups

As mentioned before, one of the principal motivations for studying profinite spaces is to study profinite *groups* in particular. So, let us define what we mean by this. First of all, we recall the definition of a topological group:

Definition 3.1. A *topological group* is a topological space G, equipped with a particular element $e \in G$ and continuous maps $\mu : G \times G \to G$, and $\iota : G \to G$, which satisfy the axioms of a *group*:

- Associativity: For every $g_1, g_2, g_3 \in G$, we have $\mu(g_1, \mu(g_2, g_3)) = \mu(\mu(g_1, g_2), g_3)$.
- Identity: For every $g_1 \in G$, we have $\mu(e, g_1) = \mu(e, g_1) = g_1$
- Inversion: For every $g_1 \in G$, we have $\mu(g_1, \iota(g_1)) = \mu(\iota(g_1), g_1) = e$.

A *morphism* of topological groups is a continuous map $f : (G_1, e_1, \mu_1, \iota_1) \to (G_2, e_2, \mu_2, \iota_2)$ which is also a group homomorphism:

- f preserves identity: $f(e_1) = e_2$, and
- f respects multiplication: $f(\mu_1(g, h)) = \mu_2(f(g), f(h))$.

Since these two properties of continuous maps are preserved by composition and satisfied by the identity, we see that topological groups form a *category*, which we call **TGrp**.

Now, we can immediately define a profinite group:

Definition 3.2. A *profinite group* is a topological group G that is profinite as a topological space. That is, G is Hausdorff, quasi-compact, and totally disconnected; equivalently, G is homeomorphic to an limit, in **Top**, of a cofiltered diagram of finite, discrete topological spaces A_i.

That is, if $\mathrm{sp}(G)$ is the underlying topological space of G, forgetting our group operations and thinking of it as an object of **Top**, it is a limit of finite discrete spaces:

$$\begin{tikzcd} & & \mathrm{sp}(G) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ \cdots \arrow[r] & A_i \arrow[r] & A_j \arrow[r] & A_k \arrow[r] & A_l \end{tikzcd}$$

Now, our definition uses only topological properties of G. However, we can show the following lemma:

Lemma 3.3. A topological group G is profinite if and only if it is isomorphic to a limit, in **TGrp**, of a cofiltered diagram of finite groups G_i, where each finite group has the discrete topology:

$$\begin{tikzcd} & & G \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ \cdots \arrow[r] & G_i \arrow[r] & G_j \arrow[r] & G_k \arrow[r] & G_l \end{tikzcd} .$$

It is not immediate that this is equivalent to definition **??**. One direction is clear: if G is such a limit in **TGrp**, then it is also a limit of the same diagram in **Top**, after forgetting the group structure.[2] Then $\mathrm{sp}(G)$ must be a profinite space, since it is a limit of a cofiltered diagram of finite discrete spaces.

The other direction will necessarily take some knowledge of group theory, and is skippable for those unfamiliar with the field. We will follow the proof in [**?**].

[2] This is not tautological, but it is true because the forgetful functor **Tgrp** $\to$ **Grp** is a right adjoint, and therefore preserves limits.

Assume that G is a topological group such that $\mathrm{sp}(G)$ is profinite. Then G is in particular locally compact and completely disconnected, by [?], TG III, §4, no. 6, every neighborhood of e in G contains an open subgroup of U of G (that is, an open subset of G which is closed under multiplication and inversion). Then the conjugates gUg^{-1} of this subgroup are also open, and they cover G; because G is compact, there must be only a finite number of conjugates: that is, every open subgroup has finite index in G. For any such open subgroup U of G, let $V = \bigcap_{g \in G} gUg^{-1}$ be the intersection of the conjugates of U; then V is also an open subgroup of finite index, and further, it is normal in G. Such normal open subgroups of finite image V also form a base for the neighborhoods of e. Therefore, we have the following diagram of finite groups for any such open normal subgroups, where $V_i \trianglelefteq V_j \trianglelefteq V_k \trianglelefteq G$:

$$\cdots \longrightarrow G/V_i \longrightarrow G/V_j \longrightarrow G/V_k \longrightarrow G,$$

where the maps are the canonical homomorphisms, and each of the quotient groups G/V is finite and has the discrete topology. In fact, this diagram is cofiltered: because the V in question form a basis for the neighborhoods of e, the intersection of any two V_i, V_j will contain a third V_k, so that we have $G/V_k \to G/V_i$ and $G/V_k \to G/V_j$.

Let G' be the limit, in **TGrp**, of this cofiltered diagram of topological groups: then it only remains to see that, in fact, G and G' are continually isomorphic. We have canonical maps $G \to G/V_i$, which are compatible with the homomorphisms between quotients, so by the universal property of the limit, there is a canonical map from G to G'. This map is dense and injective ([?], p.2), and because the groups in question are compact, it is an isomorphism. Therefore, the given definition of profinite group is equivalent to the requirement that a group be the limit of a cofiltered diagram of finite groups with the discrete topolgy.

4 An example: $\mathbb{Z}_p$

Now that we know that profiniteness is equivalent to being a limit of finite objects, in both **Top** and **TGrp**, we can use this to quickly see that certain groups are profinite. For example, let p be a prime number; then we have the additive groups $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Z}/p^3\mathbb{Z}$, and so on. We also have the projection, for any $i \geq j$, from any $\mathbb{Z}/p^i\mathbb{Z}$ to $\mathbb{Z}/p^j\mathbb{Z}$: that gives us the following diagram, which we can easily see to be cofiltered as any two elements are connected by an arrow:

$$\cdots \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}.$$

We call the limit of this diagram $\mathbb{Z}_p$, the p-adic integers. From Lemma **??**, this is a profinite group. In concrete terms, an element of $\mathbb{Z}_p$ is a list of numbers in each $\mathbb{Z}/p^k\mathbb{Z}$, each of which is congruent to the next modulo p^{k-1}. For example, if $p = 3$, then a possible list is

$$(\ldots 121, 40, 13, 4, 1),$$

where $121 \equiv 40$ modulo 81, $40 \equiv 13$ modulo 27, $13 \equiv 4$ modulo 9, and $4 \equiv 1$ modulo 3. These p-adic groups are essential to many fields of algebraic study, and they are also some of the simplest non-trivial profinite groups.

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